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A New Fuzzy Version of Euler's Method for Solving Differential Equations with Fuzzy Initial Values

(Versi Baru Kaedah Euler Kabur untuk Menyelesaikan Persamaan Pembezaan dengan Nilai-Nilai Awal Kabur)

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ABSTRACT

This paper proposes a new fuzzy version of Euler's method for solving differential equations with fuzzy initial values. Our proposed method is based on Zadeh's extension principle for the reformulation of the classical Euler's method, which takes into account the dependency problem that arises in fuzzy setting. This problem is often neglected in numerical methods found in the literature for solving differential equations with fuzzy initial values. Several examples are provided to show the advantage of our proposed method compared to the conventional fuzzy version of Euler's method proposed in the literature.

Keywords: Euler's method; fuzzy initial value; fuzzy set; optimisation

ABSTRAK

Kertas ini mencadangkan satu versi baru kaedah Euler kabur untuk menyelesaikan persamaan pembezaan dengan nilai awal kabur. Pendekatan yang digunakan adalah berasaskan kepada prinsip perluasan Zadeh dengan mengambil kira masalah kebergantungan yang wujud dalam kaedah Euler klasik. Masalah ini sentiasa diabaikan oleh penyelidik-penyelidik dalam menyelesaikan persamaan pembezaan dengan nilai awal kabur. Beberapa contoh diberikan untuk menunjukkan kelebihan kaedah yang dicadangkan dan perbandingan juga dilakukan dengan versi kabur konvensional.

Kata kunci: Kaedah Euler; nilai awal kabur; pengoptimuman; set kabur

INTRODUCTION

In modelling of real physical phenomena, differential equations play an important role in many areas of discipline, namely in economics, science and engineering. Many experts in such areas extensively use differential equations in order to make some problems under study more understandable. In many cases, information about the physical phenomena involved is always pervaded with uncertainty. According to Diniz et al. (2001), the uncertainty can arise in the experiment part, data collection, measurement process as well as when determining the initial values. These are patently obvious when dealing with "living" material, such as soil, water and microbial populations (Ahmad & De Baets 2009). Classical mathematics, however, cannot cope with this situation. Therefore, it is necessary to have some mathematical apparatus in order to understand this uncertainty. Various theories exist for describing this uncertainty and the most popular one is fuzzy set theory (Zadeh 1965).

Today, the study of differential equations with uncertainty is rapidly growing as a new area in fuzzy analysis. The terms such as "fuzzy differential equation", "fuzzy differential inclusion", and "set differential equation" are used interchangeably in referring to differential equations with fuzzy initial values or fuzzy boundary values or even differential equations dealing with functions on the space of fuzzy numbers (Buckley & Feuring 2000; Hüllermeier 1997; Kaleva 1987; Seikkala 1987; Laksmikantham 2004).

In numerical analysis, many methods have also been investigated. One of the earlier contributions was demonstrated by Ma et al. (1999). The authors proposed a fuzzy version of Euler's method to approximate the solution of fuzzy differential equations. First, the authors transformed a fuzzy differential equation by two parametric ordinary differential equations and then solved by fuzzy Euler's method. In this paper, we derived a new fuzzy version of Euler's method by taking into account the dependency problem among fuzzy sets. This problem is omitted in the numerical method proposed by Ma et al. (1999). According to Bonarini and Bontempi (1994), considering the non-dependency problem in fuzzy computation will lead to repetition of some numerical computations. Then, there exists possible errors and finally the errors may produce approximations that are wider than the correction. This is true since the preliminary results conducted by Ahmad and Hasan (2010) have shown that the solution of fuzzy differential equations obtained by using the method proposed by Ma et al. (1999) has overestimation in computation.

BASIC CONCEPTS

In this section, the basic idea of fuzzy sets will be introduced and some important concepts will be explained.

FUZZY SETS

According to Zadeh (1965), a fuzzy set is a generalisation of a classical set that allows membership function to take any value in the unit interval [0, 1]. The formal definition of a fuzzy set is as follow:

Definition 1: Let U be a universal set. A fuzzy set A in U is defined by a membership function $A(x)$ that maps every element in U to the unit interval [0, 1].

A fuzzy set A in U may also be presented as a set of ordered pairs of a generic element x and its membership value, as shown in the following equation:

$$A = \{(x, A(x)) \mid x \in U\}. \tag{1}$$

Definition 2: Let A be a fuzzy set defined in U. The support of A is the crisp set of all elements in U such that the membership function of A is non-zero, that is,

$$\text{supp}\,(A) = \{x \in U \mid A(x) > 0\}. \tag{2}$$

Definition 3: Let A be a fuzzy set defined in U. The core of A is the crisp set of all elements in U such that the membership value of A is 1, that is,

$$\text{core}\,(A) = \{x \in U \mid A(x) = 1\}. \tag{3}$$

Definition 4: Let A be a fuzzy set defined in $\Re$. A is called a fuzzy interval if

1. A is normal, that is there exists $x_0 \in \Re$ such that $A(x_0) = 1$;
2. A is convex, that is for all $x, y \in \Re$ and $0 \le \lambda \le 1$,it holds that

   $$A(\lambda x + (1 - \lambda)y) \ge \min(A(x), A(y));$$

3. A is upper semi-continuous, that is for any $x_0 \in \Re$, it holds that

   $$A(x_0) \ge \lim_{x \to x_0^{\pm}} A(x);$$

4. $[A]^0 = \overline{\{x \in \Re \mid A(x) \ge \alpha\}}$ is a compact subset of $\Re$.

Definition 5: Let A be a fuzzy interval defined in $\Re$. The α – cut of A is the crisp set $[A]^0$ that contains all elements in $\Re$ such that the membership values of A is greater than or equal to α, that is

$$[A]^\alpha = \{x \in \Re \mid A(x) \ge \alpha\}, \quad \alpha \in (0,1]. \tag{4}$$

For a fuzzy interval A, its α – cuts are closed intervals in $\Re$ and we denote them by

$$[A]^\alpha = [a_1^\alpha, a_2^\alpha], \quad \alpha \in (0,1]. \tag{5}$$

Definition 6: A fuzzy interval A is called a triangular fuzzy interval if its membership function has the following form:

$$A(x) = \begin{cases} 0, & \text{if } x < a, \\ \dfrac{x-a}{b-a}, & \text{if } a \le x \le b, \\ \dfrac{c-x}{c-b}, & \text{if } b \le x \le c, \\ 0, & \text{if } x > c, \end{cases} \tag{6}$$

and its α – cuts are simply

$$[A]^\alpha = [a + \alpha(b - a), c - \alpha(c - b)], \quad \alpha \in (0,1]. \tag{7}$$

This definition asserts that the triangular fuzzy interval A is defined by three numbers $a < b < c$, where the core of A is b and its support is the interval (a, c). Figure 1 shows the example of triangular fuzzy interval. In this paper the set of all triangular fuzzy intervals will be denoted by $F(\Re)$.

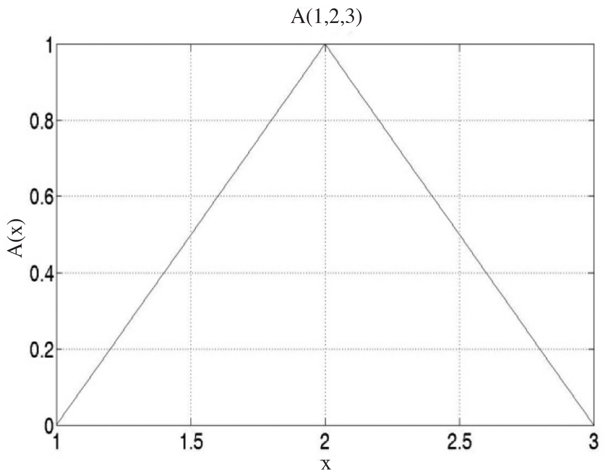

FIGURE 1. The triangular fuzzy interval A

THE EXTENSION PRINCIPLE

The idea of the extension principle is easy to understand. Let f be a function that maps from X to Y. The extension principle provides a mechanism to transform a fuzzy set defined in X to a fuzzy set defined in Y.

Let $F(X)$ and $F(Y)$ be the sets of all fuzzy sets defined in X and Y, respectively and $f : X \rightarrow Y$ be a continuous function. The function f induces a mapping $f : F(X) \rightarrow F(Y)$ such that if A is a fuzzy set in X, then its range under f is a fuzzy set B = $f(A)$ whose membership function is expressed as in the following equation (Zadeh 1975a, Zadeh 1975b and Zadeh 1975c):

$$f(A)(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)} A(x), & if\ y\in range(f),\\ 0 & ,\ if\ y\notin range(f),\end{cases} \tag{8}$$

where

$f^{-1}(y) = \{x \in X \mid f(x) = y\}$ (inverse of f)

Román-Flores et al. (2001) have shown that if $f: X \to Y$ is a continuous function, then $f: F(X) \to F(Y)$ is a well-defined function, and

$$[f(A)]^{\alpha} = f([A]^{\alpha}), \tag{9}$$

for all $\alpha \in [0,1]$ and $A \in F(X)$.

Fuzzy Initial Value Problems

In this section, we first consider the following ordinary differential equation:

$$\begin{cases}x'(t)=f(t,x(t)), & t\in[t_0,T]\\ x(t_0)=x_0\end{cases} \tag{10}$$

where f : $[t_0, T] \times \Re \to \Re$ is a continuous function defined on $[t_0, T]$ with $T > 0$ and $x_0 \in \Re$. Suppose that the initial condition in (10) is uncertain and modelled by a fuzzy interval, then we have the following fuzzy initial value problem:

$$\begin{cases}x'(t)=f(t,x(t)), & t\in[t_0,T]\\ x(t_0)=X_0\end{cases} \tag{11}$$

where $f: [t_0, T] \times F(\Re) \to F(\Re)$ is fuzzy-valued function defined on $[t_0, T]$ with $T > 0$ and $X_0 \in F(\Re)$. To interpret the connection between (10) and (11), we refer to Mizukoshi et al. (2007) and Hüllermeier (1997).

Interpretation under Zadeh's Extension Principle

Let U be an open set in $\Re$ such that there exist a solution $x(., x_0)$ of (10) with $x_0 \in U$ on the interval $[t_0, T]$ and for all $t \in [t_0, T]$, $x(t, .)$ is continuous in U. Then we can define:

$x(t, x_0) : U \to \Re$

which is the unique solution of (10). If x_0 replace by X_0, which is a fuzzy interval, then from Zadeh's extension principle we have:

$x(t, X_0) : F(U) \to F(\Re)$

which is the unique fuzzy solution of (11).

Interpretation under Hüllermeier's Approach

In agreement with Hüllermeier (1997), the differential equation in (11) can be interpreted as follow:

$$\begin{cases}x'_\beta(t)=f(t,x_\beta(t)), & t\in[t_0,T]\\ x_\beta(t_0)=[X_0]^\beta\end{cases} \tag{12}$$

where $f: [t_0,T] \times \Re \to F(\Re)$ is a real-valued function defined on $[t_0,T]$ with $T > 0$ and $\beta \in (0,1]$. For every $\beta \in (0,1]$ we say that $x_\beta : [t_0,T] \to \Re$ is the β-solution of (12) if it is absolutely continuous and satisfies (12) almost everywhere on $[t_0,T]$ with $T > 0$. Let M_β be the set of all β-solution of (12) and we define the attainable sets as:

$A_\beta(t) = \{x_\beta(t) \mid x_\beta(.) \in M_\beta\}$,

which is the β-cut of fuzzy attainable set $A(t)$. Hence, the fuzzy attainable set is the solution of (11). Chalco-Cano and Román-Flores (2008) have proven that the solution obtained by Zadeh's extension principle coincides with the solution obtained by Hüllermeier's approach.

A New Fuzzy Version of Euler's Method

First, we recall Taylor's Theorem in order to derive the classical Euler method. Suppose that $x(t)$, the unique solution of (10) have two continuous derivatives on the interval $[t_0,T]$, so that for each $j = 0,1,2,\ldots, N-1$,

$$x(t_{j+1})=x(t_j)+(t_{j+1}-t_j)y'(t_j)+\frac{(t_{j+1}-t_j)^2}{2}y''(\xi_j) \tag{13}$$

for some numbers $\xi_j \in (t_j, t_{j+1})$. By setting $h = t_{j+1} - t_j$, we have that

$$x(t_{j+1})=x(t_j)+hx'(t_j)+\frac{h^2}{2}x''(\xi_j) \tag{14}$$

and, since $x(t)$ satisfies the problem (10), we have

$$x(t_{j+1})=x(t_j)+hf(t_j,x(t_j))+\frac{h^2}{2}x''(\xi_j) \tag{15}$$

By truncating the reminder term and denoting $x_j \approx x(t_j)$, then we have the following Euler method for the problem (10):

$$x_{j+1} = x_j + hf(t_j, x_j), \tag{16}$$

for each $j = 0,1,2,\ldots, N-1$.

In order to extend the classical Euler method (16) in fuzzy setting, we need to take into account the dependency problem among fuzzy sets. First, let us consider the following situation:

$$M(h, t,x) = x+ hf(t, x). \tag{17}$$

If $x \in F(\Re)$, then (17) can be extended in fuzzy setting as follow:

$$M(h,t,X)(z)=\begin{cases}\sup\limits_{x\in M^{-1}(h,t,z)} X(x) & ,\text{if } z\in \text{range}(M),\\ 0 & ,\text{if } z\notin \text{range}(M).\end{cases} \tag{18}$$

Since (18) has very complicated structure, then we can solve it by using the α – cut of fuzzy inteval X. Let $[X]^\alpha = [x_1^\alpha, x_2^\alpha]$ be the α – cuts of X for all $\alpha \in (0,1]$, then (18) can be computed as follows:

$$M\left(h,t,[X]^\alpha\right) = \left[\min\left\{M\left(h,t,x\right) \mid x \in \left[x_1^\alpha, x_2^\alpha\right]\right\},\ \max\left\{M\left(h,t,x\right) \mid x \in \left[x_1^\alpha, x_2^\alpha\right]\right\}\right]. \tag{19}$$

By using this idea, we calculate the Euler method (18) as follows:

$$x_{j+1,1}^\alpha = \min\left\{\left(x + hf\left(t,x\right)\right) \mid x \in \left[x_{j,1}^\alpha, x_{j,2}^\alpha\right]\right\}$$

$$x_{j+1,2}^\alpha = \max\left\{\left(x + hf\left(t,x\right)\right) \mid x \in \left[x_{j,1}^\alpha, x_{j,2}^\alpha\right]\right\}.$$

To solve the minimum and maximum problems, we adopt a computational method proposed by Ahmad et al (2010). The method is described in the next section.

The Computational Method

Let $X = (a,b,c)$ be a triangular fuzzy interval. The α – cut of X is denoted by $[X]^\alpha = [x_1^\alpha, x_2^\alpha]$ for $\alpha \in (0,1]$. First, we discretise α in the form $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \alpha_n$, where $\alpha_0 = 0$ and $\alpha_n = 1$. The discretised α are equally spaced, that is $\alpha_1 = \alpha_0 + i\Delta h$, for $i = 0,1,2,\ldots, n$ and $\Delta h = \frac{1}{n} > 0$. In this study, Δh is called the discretisation spacing. After discretisation, we have a set of α with $(n + 1)$ elements:

$$\alpha = \{\alpha_0, \ldots, \alpha_i, \ldots, \alpha_n\}. \tag{20}$$

This leads to a set I of $(n + 1)$ closed intervals:

$$I = \left\{[X]^{\alpha_0}, \ldots, [X]^{\alpha_i}, \ldots, [X]^{\alpha_n}\right\} \tag{21}$$

For the different α – cuts of X the following property holds:

$$[X]^{\alpha_{i+1}} \subseteq [X]^{\alpha_i}, \quad \forall \alpha_i, \alpha_{i+1} \in [0.1] \text{ with } \alpha_i \le \alpha_{i+1}. \tag{22}$$

for $i = 0,2,\ldots, n - 1$. From (22), it is clear that the α – cuts of A at α_{i+1} is subset of the α – cuts of A at α_i (see Figure 2).

Since this property true for all $\alpha \in [0,1]$, the α – cuts of X can be constructed as the union of sub-intervals as shown in the following equation:

$$[X]^{\alpha_i} = \left[x_1^{\alpha_i}, x_1^{\alpha_{i+1}}\right] \cup \left[x_1^{\alpha_{i+1}}, x_2^{\alpha_{i+1}}\right] \cup \left[x_2^{\alpha_{i+1}}, x_2^{\alpha_i}\right]. \tag{23}$$

In order to find the numerical solution of (11), we compute $B = m(h, t, X)$ at each level of α_i for $i = 0,1,2,\ldots, n$ according to the following equations:

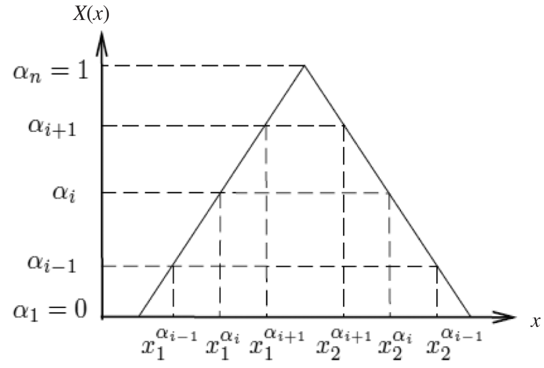

FIGURE 2. α – discretisation of a fuzzy interval

$$b_1^{\alpha_i} = \min\left[\min_{x \in \left[x_1^{\alpha_i}, x_1^{\alpha_{i+1}}\right]} m\left(t,h,x\right),\ \min_{x \in \left[x_1^{\alpha_{i+1}}, x_2^{\alpha_{i+1}}\right]} m\left(h,t,x\right),\ \min_{x \in \left[x_2^{\alpha_{i+1}}, x_2^{\alpha_i}\right]} m\left(h,t,x\right)\right], \tag{24}$$

$$b_2^{\alpha_i} = \max\left[\max_{x \in \left[x_1^{\alpha_i}, x_1^{\alpha_{i+1}}\right]} m\left(h,t,x\right),\ \max_{x \in \left[x_1^{\alpha_{i+1}}, x_2^{\alpha_{i+1}}\right]} m\left(h,t,x\right),\ \max_{x \in \left[x_2^{\alpha_{i+1}}, x_2^{\alpha_i}\right]} m\left(h,t,x\right)\right], \tag{25}$$

Here $b_1^{\alpha_i}$ and $b_2^{\alpha_i}$ are the lower and upper bounds of B, respectively at α_i for $i = 0,1,\ldots, n$. In order to interpolate the points $(b_1^{\alpha_i}, \alpha_i)$ and $(b_2^{\alpha_i}, \alpha_i)$ for all $i = 0,1,\ldots, n$, we use linear splin interpolation. Finally, a fuzzy interval B is obtained. This process is repeated for all $t_j \in [t_0, T]$ for $j = 1,2,\ldots, N - 1$.

Numerical Examples

In this section, we provide two numerical examples to show the effectiveness of our proposed method compared to the conventional fuzzy version of Euler's method proposed by Ma et al. (1999).

Example 1: Consider the following differential equation:

$$u'(t) = u(1 - 2t), \quad t \in [0,2]$$

and the fuzzy initial value is given by:

$$u_0\left(s\right) = \begin{cases} 0 & , \quad if\ s < -1/2 \\ 1 - 4s^2 & , \quad if\ -1/2 \le s \le 1/2 \\ 0 & , \quad if\ s > 1/2 \end{cases}$$

The exact solution is (see Figure 3):

$$\left[u\left(t\right)\right]^\alpha = \left[\left(-\frac{\sqrt{1-\alpha}}{2}\right)e^{-t(-1+t)}, \left(\frac{\sqrt{1-\alpha}}{2}\right)e^{-t(-1+t)}\right].$$

To get the approximate solution, we divide the interval [0,2] into 20 equally spaced subintervals. Then, we proceed with the numerical method proposed in this paper. The obtained results are plotted in Figure 4. From the graph, we can see that the approximate solution converges to the exact solution. The local errors between the approximate and exact solutions at $t_N = 2$ are given in Table 1.

In contradict to the method proposed by Ma et al. (1999), the approximate solution does not converge to the exact solution (see Figure 5). It is diverging as t increases. This shows that the method proposed by Ma et al. (1999) has overestimation in computation. This is always the case when we consider the same fuzzy interval as independent in fuzzy interval computations (see Equation 17).

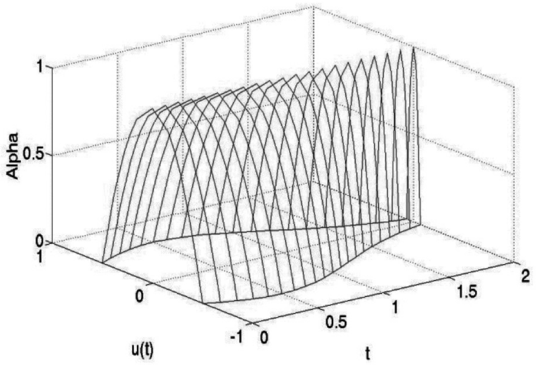

FIGURE 3. The exact solution

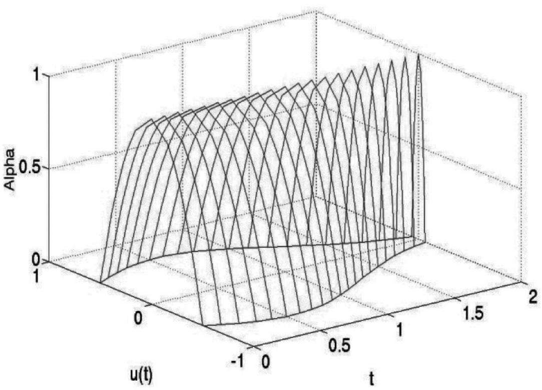

FIGURE 4. The approximate solution with $h = 0.1$ and $N = 20$

Next, we study a non-linear differential equation with fuzzy initial value.

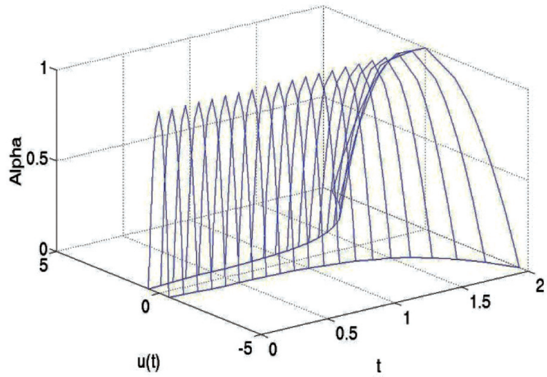

FIGURE 5. The approximate solution obtained by using the method proposed by Ma et al. (1999) with $h = 0.1$ and $N = 20$

Example 2. Consider the following non-linear differential equation:

$$u'(t) = \cos(tu), \quad t \in [0,3]$$

and the fuzzy initial value is given by:

$$u_0(s) = \begin{cases} 0 & , \quad if\ s < 0, \\ 2s/\pi & , \quad if\ 0 \le s \le \pi/2, \\ -2s/\pi + 2 & , \quad if\ \pi/2 \le s \le \pi, \\ 0 & , \quad if\ s > \pi, \end{cases}$$

Since the exact solution cannot be found analytically, we need a numerical method to approximate its solution.

TABLE 1. The local errors at $t_N = 2$

α	$u^{\alpha}_{1,2}$	$u_1^{\alpha}(2)$	Error	$u_2^{2\alpha}$	$u_2^{\alpha}(2)$	Error
0.0	-0.0644	-0.0677	0.0033	0.0644	0.0677	0.0033
0.1	-0.0611	-0.0642	0.0031	0.0611	0.0642	0.0031
0.2	-0.0576	-0.0605	0.0029	0.0576	0.0605	0.0029
0.3	-0.0539	-0.0566	0.0027	0.0539	0.0566	0.0027
0.4	-0.0499	-0.0524	0.0025	0.0499	0.0524	0.0025
0.5	-0.0456	-0.0478	0.0022	0.0456	0.0478	0.0022
0.6	-0.0407	-0.0428	0.0021	0.0407	0.0428	0.0021
0.7	-0.0353	-0.0371	0.0018	0.0353	0.0371	0.0018
0.8	-0.0288	-0.0303	0.0015	0.0288	0.0303	0.0015
0.9	-0.0204	-0.0214	0.0010	0.0204	0.0214	0.0010
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

As in the Example 1, we divide the interval [0, 3] into 20 equally spaced subintervals and proceed with the numerical method proposed in this paper. The final results are depicted in Figure 6. From the graph, we can see that the approximate solution has a periodic solution as t increases. In contrast, applying the numerical method proposed by Ma et al. (1999) the approximate solution again has diverging solution (see Figure 7).

Now the question arises: which one of the approximate solutions represents the differential equation above? The

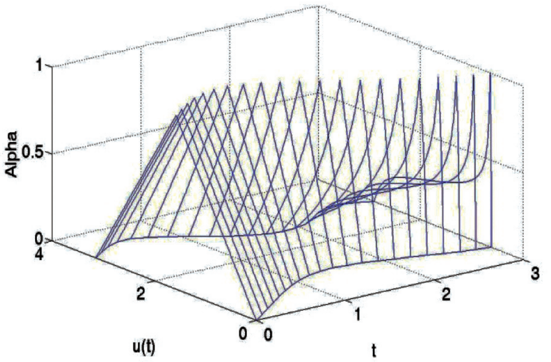

FIGURE 6. The approximate solution with $N = 20$ and $h = 0.15$

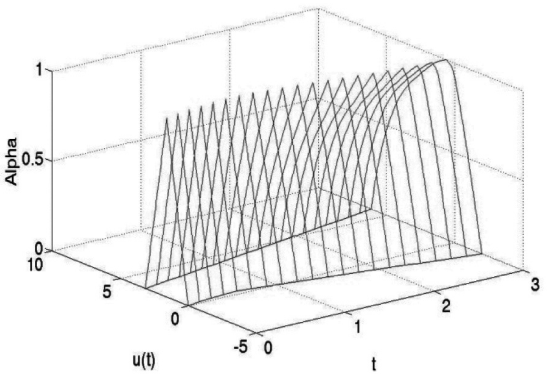

FIGURE 7. The approximate solution obtained by using the method proposed by Ma et al. (1999) with $h = 0.15$ and $N = 20$

only way to check this is by sketching a direction field, a way of predicting the qualitative behaviour of solution of a differential equation. The direction field represents the slope of approximate solution in the tu-plane. It is represented by a collection of narrow lines or tiny lines. From the practical point of view, if the approximate solution follows the direction field, then the approximate solution is the solution to the differential equation.

From the numerical results, if we look at Figure 8, the slope of the approximate solution obtained by our proposed method follows the direction field. Conversely, the approximate solution obtained by using the method proposed by Ma et al. (1999) does not follow the direction field. This is enough to prove that our proposed method

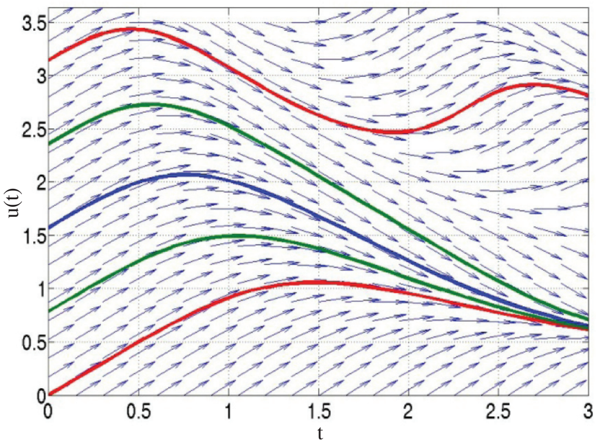

FIGURE 8. The narrow lines show the direction field of $u'(t) = \cos(tu)$. The solid curves denote the approximate solution of $u'(t) = \cos(tu)$ with $u(0) = (\pi\alpha/2, \pi - \pi\alpha/2)$ at $\alpha = 0, 0.5, 1$ generated by our proposed method, using a step size of $h = 0.15$

produced better solution.

CONCLUSION

In this paper, we have studied the numerical solution of differential equations with fuzzy initial values. By taking into account the dependency problem in fuzzy computation, we proposed a new version of Euler method, which is a generalisation of the conventional one. In order to show the capability of the proposed method, we conducted several numerical examples including linear and linear differential equations with fuzzy initial values. Final results showed that the numerical method proposed in this paper produced better solutions compared to the numerical method proposed in the literature.

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